King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 201 - Term 162 - Exam I

Duration: 120 minutes

Key

Name:____________________ ID Number:____________________

Section Number:____________________ Serial Number:____________________

ClassTime: ____________________ Instructor's Name:____________________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and legibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 8 pages of problems (Total of 10 Problems)

Question Number	Points	Maximum Points
1		9
2		14
3		18
4		14
5		8
6		6
7		9
8		9
9		8
10		5
Total		100

1. Let C be the parametric curve $x = \sin 2t$, $y = \sin t - \cos t$, $0 \le t \le \dfrac{3\pi}{4}$.

   a) (4-points) Eliminate the parameter to find a Cartesian equation of the curve C and identify it.

$y^2 = \sin^2 t - 2\sin t\cos t + \cos^2 t$ 1 pt

$y^2 = 1 - \sin 2t$ 1 pt

$y^2 = 1 - x$ 1 pt

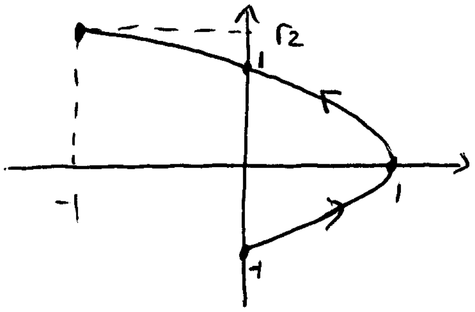
This is a parabola. 1 pt

   b) (5-points) Sketch the curve C and indicate the initial point, the terminal point, and the direction in which C is sketched as the parameter increases.

t	x	y
0	0	-1
$\pi/4$	1	0
$\pi/2$	0	1
$3\pi/4$	-1	$\sqrt{2}$

$\rightsquigarrow$ initial point is $(0,-1)$ 1 pt

$\rightsquigarrow$ terminal point is $(-1,\sqrt{2})$ 1 pt



Sketch 2 pts

direction 1 pt

2. Let C be the parametric curve $x = 3 - t^3$, $y = 2t^2$, $-\infty < t < \infty$.

a) ($\cancel{5}$ 6-points) Find a Cartesian equation of the tangent line to the curve C at the point corresponding to the parameter $t = -1$.

$\frac{dy}{dt} = 4t$ $\underline{1pt}$ $\quad \frac{dx}{dt} = -3t^2$ $\underline{1pt}$ $\quad$ slope $= \left.\frac{dy/dt}{dx/dt}\right|_{t=-1}$ $\underline{1pt}$ $= \left.\frac{4t}{-3t^2}\right|_{t=-1} = \frac{4}{3}$ $\underline{1pt}$

point corresponding to the parameter is $(x(-1), y(-1)) = (4,2)$ $\underline{1pt}$

equation of the tangent line: $(y-2) = \frac{4}{3}(x-4)$ $\underline{1pt}$

or

$y = \frac{4x}{3} + \frac{10}{3}$

b) (5-points) Find the points, if they exist, on the curve C at which there exists a vertical tangent line or a horizontal tangent line.

$\frac{dy}{dt} = 4t = 0 \Rightarrow t = 0$ $\quad \frac{dx}{dt} = -3t^2 = 0 \Rightarrow t = 0$. So $t = 0$ $\underline{1pt}$

$\lim_{t \to 0} \frac{dy/dt}{dx/dt} = \lim_{t\to 0} \frac{4t}{-3t^2} = DNE$ $\underline{1pt}$

at $t = 0$, there exists a vertical tangent line. the point corresponding to $t = 0$ is $(3,0)$. $\underline{2pt}$

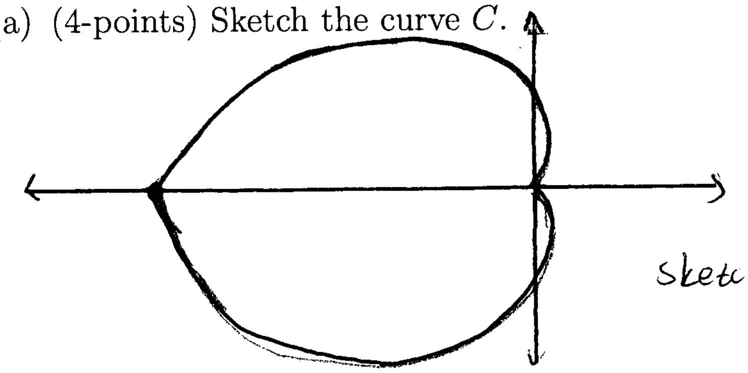
the curve C does not have horizontal tangent line. $\underline{1pt}$

c) ($\cancel{4}$ 3-points) Set up an integral that represents the area of the surface obtained by rotating the portion of the curve C on the interval $1 \le t \le 2$ about the x-axis.

$$\underbrace{\int_1^2 2\pi}_{1pt} \; \underbrace{2t^2}_{1pt} \; \underbrace{\sqrt{16t^2 + 9t^4}\, dt}_{1pt}$$

3. Let C be the polar curve $r = \theta^2$, $-\pi \le \theta \le \pi$.

(a) (4-points) Sketch the curve C.



sketch $\underline{2 \text{ pts}}$

θ	r
$-\pi$	π^2
$-\pi/2$	$\pi^2/4$
0	0
$\pi/2$	$\pi^2/4$
π	π^2

important points $\underline{2 \text{ pts}}$

(b) (5-points) Find the slope of the tangent to the curve C at $\theta = \pi/4$.

$\frac{dr}{d\theta} = 2\theta$ $\underline{1 \text{pt}}$

$$\text{slope} = \left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \left.\frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}\right|_{\theta=\pi/4} = \left.\frac{2\theta\sin\theta + \theta^2\cos\theta}{2\theta\cos\theta - \theta^2\sin\theta}\right|_{\theta=\pi/4}$$

$\underline{2 \text{pts}}$ $\underline{1 \text{pt}}$

$$= \left.\frac{2+\theta}{2-\theta}\right|_{\theta=\frac{\pi}{4}} = \frac{2+\pi/4}{2-\pi/4} \quad \text{or} \quad \frac{8+\pi}{8-\pi}$$

$\underline{1 \text{pt}}$

(c) (9-points) Find the length of the curve C.

$$\text{Length} = \int_{-\pi}^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{-\pi}^{\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta$$

$\underline{2 \text{pts}}$ $\underline{1 \text{pt}}$

$$= \int_{-\pi}^{\pi} |\theta| \sqrt{\theta^2 + 4}\, d\theta = 2\int_0^{\pi} \theta\sqrt{\theta^2+4}\, d\theta$$

$\underline{2 \text{pts}}$

$\underline{\text{do subs}}$

$u = \theta^2 + 4$

$du = 2\theta\, d\theta$

$\theta = 0 \Rightarrow u = 4$

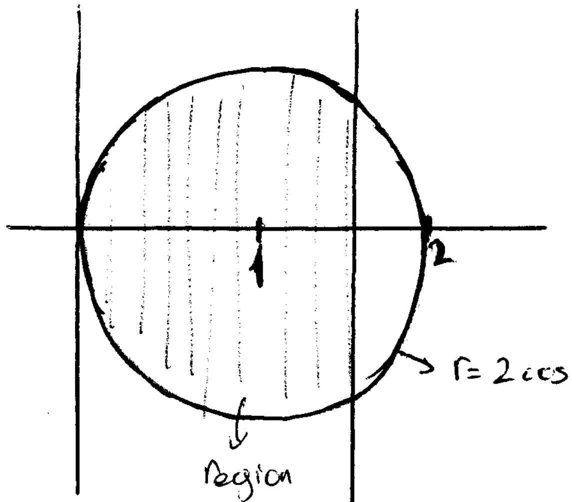
$\theta = \pi \Rightarrow u = \pi^2 + 4$

$\underline{1 \text{pt}}$

$$= \int_4^{\pi^2+4} \sqrt{u}\, du = \frac{2}{3} u^{3/2}\Big|_4^{\pi^2+4} = \frac{2}{3}\left[(\pi^2+4)^{3/2} - 8\right]$$

$\underline{1 \text{pt}}$ $\underline{1 \text{pt}}$ $\underline{1 \text{pt}}$

4. (14-points) Let R be the region inside the circle $r = 2\cos\theta$ and between the vertical lines $\theta = \dfrac{\pi}{2}$ and $r = \dfrac{3}{2}\sec\theta$. Sketch the region R and find its area.



$r = 2\cos\theta$. $\underline{1pt}$

Region $\underline{1pt}$

$\theta = \pi/2$

$r = \frac{3}{2}\sec\theta$ $\underline{1pt}$

$\underline{\text{intersection}}$

$$2\cos\theta = \frac{3}{2}\sec\theta$$

$$\cos^2\theta = \frac{3}{4}$$

$$\cos\theta = \frac{\sqrt{3}}{2}$$

$$\theta = \pm \pi/6. \quad \underline{1pt}$$

$$\text{Area} = \underset{\underline{1pts}}{2}\left[\underbrace{\frac{1}{2}\int_0^{\pi/6} \frac{9}{4}\sec^2\theta\, d\theta}_{\underline{2pts}} + \underbrace{\frac{1}{2}\int_{\pi/6}^{\pi/2} 4\cos^2\theta\, d\theta}_{\underline{2pts}}\right]$$

$$= \frac{9}{4}\int_0^{\pi/6} \sec^2\theta\, d\theta + 4\int_{\pi/6}^{\pi/2} \cos^2\theta\, d\theta$$

$$= \frac{9}{4}\int_0^{\pi/6} \sec^2\theta\, d\theta + 2\int_{\pi/6}^{\pi/2} \cos 2\theta + 1\, d\theta \qquad \underline{1pt}$$

$$= \frac{9}{4}\tan\theta\Big|_0^{\pi/6} + (\sin 2\theta + 2\theta)\Big|_{\pi/6}^{\pi/2} \qquad \underline{2pts}$$

$$= \frac{9}{4}\left(\frac{1}{\sqrt{3}} - 0\right) + (\sin\pi + \pi) - (\sin\pi/3 + \pi/3)$$

$$= \frac{\sqrt{3}}{4} + \frac{2\pi}{3}$$

$\underline{1pt}$

5. Consider the sphere given by the equation $x^2 + y^2 + z^2 - 2x - 4y + 8z = 15$.

   (a) (4-points) Find its center and radius.

$$x^2 - 2x + 1 + y^2 - 4y + 4 + z^2 + 8z + 16 = 15 + 1 + 4 + 16$$

$$(x-1)^2 + (y-2)^2 + (z+4)^2 = 36 \qquad \underline{2\,pts}$$

center $(1, 2, -4)$ radius 6

$\underline{1\,pt}$ $\underline{1\,pt}$

   (b) (4-points) Determine whether the origin is located outside, on, or inside this sphere. Give reasons to your answer **without** using sketches.

distance between the center and the ~~radius~~ origin is $\sqrt{1+4+16} = \sqrt{21}$ $\underline{2\,pts}$

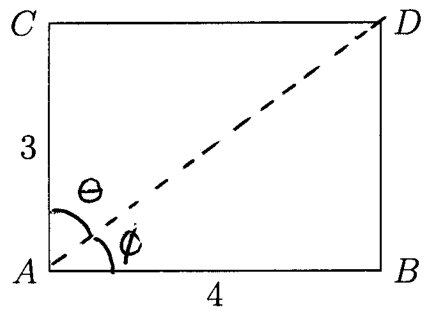
as $\underbrace{\sqrt{21} < 6}_{1\,pt}$, where 6 is the radius, $\underbrace{\text{the origin is located inside.}}_{1\,pt}$

6. (6-points) Describe in terms of inequalities, the closed solid cylinder between the disk with center $(1, -2, -1)$ and radius 2 on the plane $y = -2$ and the disk with center $(1, 3, -1)$ and radius 2 on the plane $y = 3$.

$$\underbrace{x^2 + z^2 \leq 4}_{3\,pts}, \quad \text{and} \quad \underbrace{-2 \leq y \leq 3}_{3\,pts},$$

$$(x-1)^2 + (z+1)^2 \leq 4$$

7. Let A, B, C, and D be the vertices of the rectangle whose sides are 3 and 4 units as shown in the picture



(a) (3-points) Express the vector operation $\overrightarrow{AC}+\overrightarrow{BD}-\overrightarrow{BC}$ as a single vector and find its magnitude.

$$\overrightarrow{AC}+\overrightarrow{BD}-\overrightarrow{BC} = \overrightarrow{AC}+\overrightarrow{CB}+\overrightarrow{BD} \quad 1pt$$

$$= \overrightarrow{AD} \quad 1pt$$

$$|\overrightarrow{AD}| = 5 \quad 1pt$$

(b) (3-points) Find $\overrightarrow{AC} \bullet \overrightarrow{AD}$.

$$\overrightarrow{AC} \bullet \overrightarrow{AD} = |\overrightarrow{AC}|\,|\overrightarrow{AD}| \cos\theta \quad 1pt$$

$$= |\overrightarrow{AC}|\,|\overrightarrow{AD}| \cdot \frac{|\overrightarrow{AC}|}{|\overrightarrow{AD}|} \quad 1pt$$

$$= |\overrightarrow{AC}|^2 = 9 \quad 1pt$$

(c) (3-points) Find $|\overrightarrow{AB} \times \overrightarrow{AD}|$.

$$|\overrightarrow{AB} \times \overrightarrow{AD}| = |\overrightarrow{AB}|\,|\overrightarrow{AD}| \cdot \sin\phi \quad 1pt$$

$$= |\overrightarrow{AB}|\,|\overrightarrow{AD}| \cdot \frac{|\overrightarrow{BD}|}{|\overrightarrow{AD}|} \quad 1pt$$

$$= |\overrightarrow{AB}|\,|\overrightarrow{BD}| = 4 \cdot 3 = 12 \quad 1pt$$

8. Let $\mathbf{a} = \langle 1, -2, 2\rangle$ and $\mathbf{b} = \langle 4, 1, 5\rangle$ be two vectors.

(a) (4-points) Find $\text{proj}_{\mathbf{a}}\mathbf{b}$.

$a \cdot b = 4 - 2 + 10 = 12$ 1pt

$|a| = \sqrt{1 + 4 + 4} = 3$ 1pt

$\text{proj}_a b = \frac{a \cdot b}{|a|^2} a = \frac{12}{9}\langle 1, -2, 2\rangle$ or $\langle \frac{4}{3}, \frac{-8}{3}, \frac{8}{3}\rangle$

1pt 1pt

(b) (3-points) Verify that $\mathbf{b} - \text{proj}_{\mathbf{a}}\mathbf{b}$ is orthogonal to $\mathbf{a}$.

$b - \text{proj}_a b = \langle 4, 1, 5\rangle - \langle \frac{4}{3}, \frac{-8}{3}, \frac{8}{3}\rangle$

$= \langle \frac{8}{3}, \frac{11}{3}, \frac{7}{3}\rangle$ 1pt

$b - \text{proj}_a b$ is orthogonal to a if and only if $(b - \text{proj}_a b) \cdot a = 0$

1pt

$(b - \text{proj}_a b) \cdot a = \langle \frac{8}{3}, \frac{11}{3}, \frac{7}{3}\rangle \cdot \langle 1, -2, 2\rangle = \frac{8}{3} - \frac{22}{3} + \frac{14}{3} = 0$

1pt

(c) (2-points) Express $\mathbf{b}$ as sum of two vectors where one of the vectors is parallel to $\mathbf{a}$ and the other is orthogonal to $\mathbf{a}$.

a and $\text{proj}_a b$ are parallel. 1pt

Then $b = \text{proj}_a b + (b - \text{proj}_a b)$

or 1pt

$\langle 4, 1, 5\rangle = \langle \frac{4}{3}, \frac{-8}{3}, \frac{8}{3}\rangle + \langle \frac{8}{3}, \frac{11}{3}, \frac{7}{3}\rangle$

9. (8-points) Let $P(2,0,-3)$, $Q(1,4,5)$, and $R(7,2,9)$ be the vertices of a triangle. Find the angle at the vertex Q.

$\overrightarrow{QP} = \langle 1, -4, -8 \rangle$ 1pt $\Rightarrow$ $|\overrightarrow{QP}| = \sqrt{1+16+64} = 9$ 1pt

$\overrightarrow{QR} = \langle 6, -2, 4 \rangle$ 1pt $\Rightarrow$ $|\overrightarrow{QR}| = \sqrt{36+4+16} = \sqrt{56}$ 1pt

$\overrightarrow{QP} \cdot \overrightarrow{QR} = 6 + 8 - 32 = -18$ 1pt

Let θ be the angle at the vertex Q. Then

$$\cos\theta = \frac{\overrightarrow{QP} \cdot \overrightarrow{QR}}{|\overrightarrow{QP}|\,|\overrightarrow{QR}|} = \frac{-18}{9 \cdot \sqrt{56}} \Rightarrow \theta = \cos^{-1}\left(\frac{-1}{\sqrt{14}}\right)$$

1pt 1pt 1pt

10. (5-points) Find the value of x so that the volume of the parallelepiped determined by the vectors $\mathbf{u} = 5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 4\mathbf{i} - \mathbf{j} + \mathbf{k}$, and $\mathbf{w} = \mathbf{i} - \mathbf{j} + x\mathbf{k}$ is equal to 0.

$$\text{Volume} = |(\vec{u} \times \vec{v}) \cdot \vec{w}|$$

$$= \left| \begin{vmatrix} 5 & -2 & 1 \\ 4 & -1 & 1 \\ 1 & -1 & 2 \end{vmatrix} \right| = 0 \qquad \text{2pt}$$

x

Then $5\begin{vmatrix} -1 & 1 \\ -1 & x \end{vmatrix} + 2\begin{vmatrix} 4 & 1 \\ 1 & x \end{vmatrix} + \begin{vmatrix} 4 & -1 \\ 1 & -1 \end{vmatrix} = 0$

$\Rightarrow 5(-x+1) + 2(4x-1) + (-3) = 0$ 2pt

$\Rightarrow 3x = 0$

$\Rightarrow x = 0$ 1pt