MATH 102 (Semester 102)
Review One: Series

1. *Basic Definitions:*

   - A series $\sum_{i=1}^{\infty} a_i$ has two associated sequences:

     (a) The **underlying sequence**:

     $$a_1, a_2, a_3, \ldots$$

     (b) The **sequence of partial sums**: $s_1, s_2, s_3, \ldots$ where

     $$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots a_{n-1} + a_n.$$

   - The infinite sum is defined to be:

     $$\sum_{i=1}^{\infty} a_i = \lim_n s_n = \lim_n \sum_{i=1}^{n} a_i.$$

     If this limit exists, the series is said to **converge**. If it does not exist, the limit is said to **diverge**.

   - **Divergence Test**
     If the underlying sequence does not converge to 0, then the series diverges. That is, if

     $$\lim_n a_n \neq 0,$$

     then

     $$\sum_{n=1}^{\infty} a_n \text{ diverges .}$$

     Note: the opposite is NOT true. There are series with the property that $\lim_n a_n = 0$ but $\sum_n^{\infty} a_n$ diverges. The most famous is the **harmonic series**.

2. *Types of Series:*

   - **Geometric series.** A series of the form

     $$\sum_{n=1}^{\infty} ar^n$$

     is called a **geometric series**. Geometric series converge to $\frac{a}{1-r}$ if $|r| < 1$ and diverge if $|r| \geqslant 1$.

- **Telescoping series**

  The sum of these series can often be calculated exactly. See for example §11.2 Exercises 35-40.

- **p-series**

  These are series of the form
  $$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$
  They converge if and only if $p > 1$ (by the integral test).

  The most famous p-series is the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ ($p = 1$), which diverges.

- **Alternating Series**

  These are series that alternate between positive and negative numbers. That is, series of the form
  $$\sum_{n=1}^{\infty} (-1)^n b_n, \text{ or } \sum_{n=1}^{\infty} (-1)^{n+1} b_n \text{ where } b_n > 0.$$

3. *Tests for convergence:*

- **Divergence Test**

  See page 1.

- **Integral Test**

  If $f(x)$ is a continuous, positive, and decreasing function and $a_n = f(n)$ for all whole numbers, then
  $$\sum_{n=1}^{\infty} a_n \text{ converges if and only if } \int_1^{\infty} f(x)dx \text{ converges.}$$

- **Comparison Tests** There are two types of comparison test - the *Comparison Test* and the *Limit Comparison Test.* Both tests compare one series with another series.

  - **Comparison Test:**

    Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are positive series and $b_n \geqslant a_n$ for all n. Then,
    $$\text{if } \sum_{n=1}^{\infty} b_n \text{ converges, then } \sum_{n=1}^{\infty} a_n \text{ converges, and}$$
    $$\text{if } \sum_{n=1}^{\infty} a_n \text{ diverges, then } \sum_{n=1}^{\infty} b_n \text{ diverges.}$$

- **Limit Comparison Test:**
  If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are positive series, and if

$$\lim_n \frac{a_n}{b_n}$$

  converges to a number $\neq 0$ and $\neq \infty$, then either both series converge, or both series diverge.

- **Alternating Series Test**
  If $\sum_{n=1}^{\infty}(-1)^n b_n$ is an alternating series, then it converges if
  (a) $a_{n+1} \leqslant a_n$ for all n and
  (b) $\lim_n a_n = 0$.

- **Ratio and Root tests**
  A series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if it is convergent and $\sum_{n=1}^{\infty} |a_n|$ is convergent.
  A series $\sum_{n=1}^{\infty} a_n$ is *conditionally convergent* if it is convergent but $\sum_{n=1}^{\infty} |a_n|$ is divergent.
  (a) (Ratio Test)
    - If $\lim_n \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
    - If $\lim_n \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.
    - If $\lim_n \left|\frac{a_{n+1}}{a_n}\right| = 1$, then no conclusion is possible.

  (b) (Root Test)
    - If $\lim_n \sqrt[n]{|a_n|} < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
    - If $\lim_n \sqrt[n]{|a_n|} > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.
    - If $\lim_n \sqrt[n]{|a_n|} = 1$, then no conclusion is possible.

4. *Estimation of Series:*

The remainder R_n is defined as follows.

If $s = \sum_{i=1}^{\infty} a_i$, and $s_n = \sum_{i=1}^{n} a_i$, then $R_n = s - s_n$. That is

$$R_n = \sum_{i=n+1}^{\infty} a_i.$$